New exponent for dynamic correlations in domain growth

1990 J. Phys. A: Math. Gen. 23 L279

(http://iopscience.iop.org/0305-4470/23/6/008)

J. Phys. A: Math. Gen. 23 (1990) L279–L284. Printed in the UK

LETTER TO THE EDITOR

New exponent for dynamic correlations in domain growth

T J Newman and A J Bray

Department of Theoretical Physics, The University, Manchester M13 9PL, UK

Received 18 December 1989

Abstract. The dynamics of the n-component Ginzburg–Landau model with non-conserved order parameter (model A) are considered following a quench to zero temperature. The correlation function of the order parameter field is found, in the $1/n$ expansion, to have the asymptotic scaling form $C_k(t, t') = t'^{d/2}(t/t')^{\lambda/2} f(k^2 t, k^2 t')$ for $t \gg t'$, with $f(0,0) =$ constant. The new exponent λ is calculated to $O(1/n)$ for general space dimension d, and has a non-trivial dependence on n and d.

The study of the growth of order, following the quench of a system from the high-temperature to the low-temperature phase, has a long history (see e.g. [1]). Most work has been devoted to the equal-time structure factor, which carries information about the evolving order. Relatively little consideration has been given to correlations between the order parameter field at different times†. In this letter we show that such correlations require a new, non-trivial exponent for their description.

We consider the asymptotic dynamics of a system with a non-conserved vector order parameter, $\phi = (\phi^1, \ldots, \phi^n)$, following a quench to zero temperature. In particular we shall derive, via a diagrammatic expansion of the equation of motion, an expression for the correlation function $C_k(t, t') = [\phi^i_k(t)\phi^i_{-k}(t')]$. The equal-time structure factor is just $S_k(t) = C_k(t, t)$. Here square brackets indicate an average over the ensemble of possible initial conditions, and $\phi_k(t)$ is simply the spatial Fourier transform of $\phi(x, t)$. For late times, $S_k(t)$ is expected to take the scaling form [1]

$$S_k(t) = L(t)^d g(kL(t)) \tag{1}$$

where $L(t) \sim t^{1/2}$ is the characteristic scale of spatial order at time t. The prefactor $L(t)^d$ in (1) indicates the emergence of a Bragg peak at $k = 0$ for $t \to \infty$. Below we shall show, in the context of a $1/n$ expansion for a non-conserved order parameter, that the more general correlation function $C_k(t, t')$ has the scaling form

$$C_k(t, t') = L(t')^d (L(t)/L(t'))^\lambda f(kL(t), kL(t')) \qquad t \gg t' \tag{2}$$

where $f(0, 0)$ is a constant and λ is a new exponent with no simple dependence on n and d. To $O(1/n)$ it is given by

$$\lambda = d/2 - (4/3)^{d/2}(2d(d+2)/9)B(d/2+1, d/2+1)(1/n) + O(1/n^2) \tag{3}$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function [3].

A recent analysis [4] of the above system quenched to the *critical* point yields a form similar to (2), but with d replaced by $2 - \eta$ and $L(t) \sim t^{1/z}$ where z is the dynamic

† Furukawa [2] has discussed 'multi-time scaling' in general terms.

critical exponent. In this case also λ is a *new* exponent, i.e. it cannot be expressed in terms of z and the static critical exponents. We expect the $T=0$ result for λ to be valid asymptotically for a quench to any temperature $T<T_c$ since thermal fluctuations should be irrelevant below T_c for $d>2$. ($d>1$ for a scalar order parameter.) For $d<2$ our results are limited to $T=0$ since the system does not order at non-zero temperature.

The system is described by a standard Ginzburg-Landau functional, and the dynamics is governed by the following equation of motion:

$$\frac{\partial \phi_k^i}{\partial t}=(r-k^2)\phi_k^i-\frac{u}{n}\sum_{j,p,q}\phi_{k-p-q}^i\phi_p^j\phi_q^j. \tag{4}$$

Since the system is quenched to zero temperature, there is no thermal noise, and all averages will be taken over the initial conditions. These will be taken to have a Gaussian distribution, with mean zero and correlator defined by

$$[\phi_k^i(0)\phi_{-k'}^j(0)]=\Delta\delta_{i,j}\delta_{k,k'}. \tag{5}$$

For this distribution it is easy to show, using integration by parts, that $C_k(t,0)$ is trivially related to the averaged response of the order parameter to the initial conditions:

$$G_k(t)\equiv\left[\frac{\partial\phi_k^i(t)}{\partial\phi_k^i(0)}\right]=\frac{1}{\Delta}C_k(t,0). \tag{6}$$

We shall refer to $G_k(t)$ as the 'response function'.

It is simplest, in the first instance, to set the shorter time t' equal to zero, i.e. to look at the correlation of the order parameter with the initial condition, as in (6). The role of t' in (2) is then played by a suitable short-time cut-off t_0 (see below), and the exponent λ can be inferred from the t dependence. In particular, the response function has the scaling form $G_k(t)=t^{\lambda/2}h(k^2t)$. The extension to general t' is straightforward.

To proceed we perform a diagrammatic expansion about the solution of the Gaussian model ($u=0$). Setting $u=0$ in (4) gives

$$\phi_k^i(t)=\phi_k^i(0)\exp(r-k^2)t. \tag{7}$$

Expanding about this solution in powers of u and averaging over the initial conditions gives, in the limit $n\to\infty$, the following self-consistent expression for $G_k(t)$:

$$G_k(t)=\exp\left(-\int_0^t \mathrm{d}t'[k^2+A(t')]\right) \tag{8}$$

where

$$A(t)=-r+u\Delta\sum_p G_p(t)^2. \tag{9}$$

To determine $A(t)$ we substitute (8) into (9) and perform an unrestricted sum over the momentum p. This yields

$$A(t)=-r+uK_d\Delta t^{-d/2}\exp\left(-2\int_0^t \mathrm{d}t'\,A(t')\right) \tag{10}$$

where K_d is defined by $\Sigma_p\exp(-2p^2t)=K_d/t^{d/2}$. We now restrict our attention to long times. For consistency in (10) we require $A(t)\to -d/4t$ for $t\to\infty$. Then $\int_0^t \mathrm{d}t'\,A(t')\to-(d/4)\ln(t/t_0)$ for large t. Using this in (10) determines t_0 through

$$r=uK_d\Delta/t_0^{d/2}. \tag{11}$$

We then have the leading-order expressions for $G_k(t)$ and $C_k(t, t')$, (for $n \to \infty$, $C_k(t, t') = \Delta G_k(t) G_k(t')$):

$$G_k(t) = (t/t_0)^{d/4} \exp(-k^2 t) \tag{12}$$

and

$$C_k(t, t') = \Delta (tt'/t_0^2)^{d/4} \exp[-k^2(t + t')] \tag{13}$$

for $t, t' \gg t_0$. Summing $S_k(t) \equiv C_k(t, t)$ over k and using (11) yields the expected result $\Sigma_k S_k(t) = K_d \Delta / t_0^{d/2} = r/u$, equal to the square of the equilibrium magnetisation.

We notice from this leading-order calculation that the structure function has the expected scaling form (1), with $L(t) = t^{1/2}$ and $f(x) = \exp(-2x)$. This leading-order result has also been obtained by Coniglio and Zannetti [5]. The response function also scales, with $\lambda = d/2$ to leading order.

We now wish to extend this leading-order result by calculating the corrections to the above correlation functions due to terms in the diagrammatic expansion of $O(1/n)$. We shall find the following results: (i) the structure factor retains the scaling form (1) at $O(1/n)$, with a modified scaling function; (ii) the exponent λ picks up a contribution of $O(1/n)$ which is a non-trivial function of d, implying that it is distinct from established exponents. Only the outline of the calculation shall be given here; details of the complete calculation and other results will be presented in a future publication [6].

We shall denote the $O(1/n)$ contributions to the structure factor and the response function by a prime, and add an '∞' superscript to the previous leading-order results. Therefore

$$G_k(t) = G_k^{\infty}(t) + \frac{1}{n} G_k'(t) + O\left(\frac{1}{n^2}\right) \tag{14}$$

and

$$C_k(t, t') = C_k^{\infty}(t, t') + \frac{1}{n} C_k'(t, t') + O\left(\frac{1}{n^2}\right). \tag{15}$$

Keeping all $O(1/n)$ contributions from the diagrammatic expansion, we find that $G_k'(t)$ and $C_k'(t, t')$ have the following forms:

$$G_k'(t) = 2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, G_k^{\infty}(t_2) G_k^{\infty}(t, t_1) \Pi_k(t_1, t_2) \tag{16}$$

and

$$C_k'(t, t') = \Delta[G_k^{\infty}(t) G_k'(t') + G_k^{\infty}(t') G_k'(t)] + 2 \int_0^t \mathrm{d}t_1 \int_0^{t'} \mathrm{d}t_2\, G_k^{\infty}(t, t_1) G_k^{\infty}(t', t_2) \Omega_k(t_1, t_2). \tag{17}$$

The functions $\Pi_k(t_1, t_2)$ ('self-energy') and $\Omega_k(t_1, t_2)$ are expressed in terms of diagrams and are shown in figures 1 and 2 respectively.

The elements of these diagrams are as follows. A circle corresponds to the origin of time and carries a weight of Δ. A single line emerging from a circle corresponds to the response function calculated at leading order and is given by (12). There are two elements which were not present at leading order and these need to be explicitly calculated before we may evaluate the diagrams. The first of these is a single line

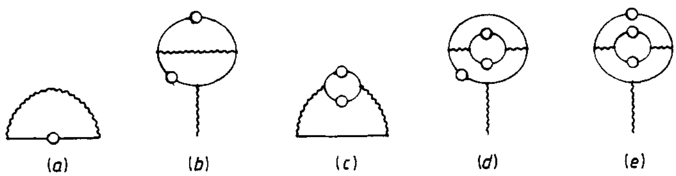

Figure 1. Diagrams contributing to the 'self-energy' $\Pi_k(t_1, t_2)$ at $O(1/n)$. Diagrams (b) and (d) each carry a combinatoric factor of 2.

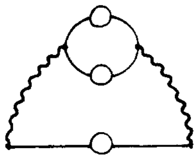

Figure 2. $O(1/n)$ diagram for the function $\Omega_k(t_1, t_2)$ of (17).

connecting two non-zero times. This is written as $G_k^\infty(t, t')$ where $t > t'$. It corresponds to the response of the order parameter (at time t) with respect to thermal noise (acting at t') in the limit of infinitesimally weak noise, i.e.

$$G_k^\infty(t, t') = \left[\frac{\delta\phi_k^i(t)}{\delta\xi_k^i(t')}\right]_{\xi=0}. \tag{18}$$

After explicit calculation, we find that

$$G_k^\infty(t, t') = \frac{G_k^\infty(t)}{G_k^\infty(t')} = (t/t')^{d/4} \exp[-k^2(t-t')] \tag{19}$$

where the last equality requires $t' \gg t_0$. The second element is the wavy line, which is a 'dressed vertex' given by the usual 'bubble sum' of the $1/n$ expansion (see e.g. [7]). It is written as $v_k(t, t')$ with $t > t'$, and satisfies the equation

$$v_k(t, t') = u\delta(t-t') - 2u\Delta \int_{t'}^{t} \mathrm{d}t''\, v_k(t, t'') \sum_p G_p^\infty(t') G_{-p}^\infty(t'') G_{k+p}^\infty(t'', t'). \tag{20}$$

Inserting (12) and (19), evaluating the momentum sum, and making use of (11), yields the integral equation

$$v_k(t, t') = u\delta(t-t') - 2r \int_{t'}^{t} \mathrm{d}t''\, v_k(t, t'') \exp\left(-k^2 \frac{(t''^2 - t'^2)}{2t''}\right). \tag{21}$$

We have been unable to solve this integral equation exactly. However, we can express $v_k(t, t')$ in terms of a controlled expansion about the solution of the soluble integral equation

$$f_k(t, t') = u\delta(t-t') - 2r \int_{t'}^{t} \mathrm{d}t''\, f_k(t, t'') \exp[-k^2(t''-t')]. \tag{22}$$

The solution of this second integral equation may be obtained by either Laplace transform, or differentiation, and is given by

$$f_k(t, t') = u\{\delta(t-t') - 2r \exp[-2r(t-t')]\} \exp[-k^2(t-t')]. \tag{23}$$

Expanding around this solution in (21) yields the following form for $v_k(t, t')$:

$$v_k(t, t') = f_k(t, t')\rho_k(t, t') \tag{24}$$

where

$$\rho_k(t, t') = 1 + \frac{k^2(t-t')^2}{2t}\left(1 + \frac{2r(t-t')}{3} + O(r^2(t-t')^2)\right) \tag{25}$$

plus terms of higher order in $k^2(t-t')^2/2t$.

The diagrams in figures 1 and 2 may now be evaluated. Since we are only interested in asymptotically large times, we exploit the fact that $rt \gg 1$ when integrating over the time arguments of the wavy line; i.e. when integrating $v_k(t, t')g(t')$ over t', for a general function $g(t')$, we have the following relation (via integration by parts):

$$\int^t \mathrm{d}t\, v_k(t, t')g(t') = \frac{1}{2r}\left(\frac{\partial}{\partial t'}[\exp - k^2(t-t')]\rho_k(t, t')g(t')\right)_{t'=t}\left(1 + O\left(\frac{1}{rt}\right)\right). \tag{26}$$

This implies that for all leading-order results, we may simply set $\rho_k(t, t') = 1$, because $\{\partial\rho_k(t, t')/\partial t'\}_{t'=t} = 0$.

In order to determine the exponent λ of (2), it is sufficient, and computationally convenient, to work at external momentum $\boldsymbol{k} = 0$, i.e. we compute $G_0(t) \sim L(t)^\lambda \sim t^{\lambda/2}$. We find that *all* diagrams in figure 1 (after integration over external propagators) have leading large-t behaviour proportional to $\ln(t/t_0)G_0^\infty(t)$, where we have introduced t_0 as a lower cut-off on the (logarithmically divergent) final time integral. However, diagrams cancel in pairs as far as the leading logarithm is concerned: $(1(a)+1(b))$ and $(1(c)+1(d))$ each give a net contribution of $O(1)$ times $G_0^\infty(t)$. This leaves diagram $1(e)$. Calculating the prefactor of the logarithmic term from $1(e)$ explicitly gives

$$G_0(t) = G_0^\infty(t)[1 + (1/n)(a\ln(t/t_0) + O(1)) + O(1/n^2)] \tag{27}$$

where a is a non-trivial function of d. Using (12) for $G_0^\infty(t)$, and exponentiating the logarithm, gives $G_0(t) \simeq (t/t_0)^{\lambda/2}$ with λ given by (3).

The analogous result for $C_0(t, t')$, to leading logarithmic accuracy, is

$$C_0(t, t') = \Delta G_0^\infty(t)G_0^\infty(t')\{1 + (a/n)[\ln(t'/t_0) + \ln(t/t_0) - 2\ln(t'/t_0)] + O(1/n^2)\} \tag{28}$$

for $t \gg t'$, where the three logarithms are associated with the three terms in (17). The equal-time correlation function is also given by (28), with $t' = t$, to leading logarithmic accuracy. For $t' = t$, therefore, the final term in (17), associated with the diagram of figure 2, exactly cancels the contributions of diagram $1(e)$ to each of the first two terms in (17). Hence the exponent associated with the structure factor is unchanged to $O(1/n)$ (and presumably to all orders), and $S_k(t)$ retains the standard scaling form (1), although the scaling function $g(x)$ does acquire corrections. For $t \gg t'$, using (12) for G_k^∞ and exponentiating the logarithms in (28) gives $C_0(t, t') \simeq (t'/t_0)^{d/2}(t/t')^{\lambda/2}$, as required by (2).

For general $\boldsymbol{k}$, the arguments of scaling functions retain the form k^2t to $O(1/n)$. We expect this to remain true to all orders in $1/n$, since the scaling variable reflects the growth of the characteristic length scale $L(t) \sim t^{1/2}$. By analogy with critical phenomena, we can think of k^2t as k^zt with $z = 2$ being the 'dynamical exponent at the zero-temperature fixed point' for a non-conserved order parameter [8].

In conclusion, a new exponent for dynamic correlations in domain growth has been evaluated: it is the analogue for the 'zero-temperature fixed point' of the recently discussed exponent describing dynamic correlations following a quench to the critical point [4]. For the latter case, also, the exponent λ can be calculated for $n=\infty$ [4]. The result is $\lambda_c=(4-d)/2$, where the subscript indicates a critical exponent. For $d=2$ we find $\lambda=\lambda_c(=1)$, as expected since the critical and zero-temperature fixed points merge as d approaches the 'lower critical dimension' $d_l=2$. Note that, while λ_c can also be evaluated, for general n, as an expansion in $\varepsilon=4-d$ [4], there is no analogous expansion for λ: the $1/n$ expansion seems to be the only systematic tool available at the $T=0$ fixed point.

The exponent λ is, as far as we are aware, the first non-trivial exponent to be predicted for a zero-temperature fixed point. It should be relatively straightforward to measure, e.g. by computer simulation, for both scalar and vector order parameters. The autocorrelation function with the initial condition, $C(t)=[\phi_i(t)\cdot\phi_i(0)]$, would seem the simplest quantity to compute numerically. Scaling gives $C(t)\sim t^{\lambda/2}\Sigma_k h(k^2t)\sim 1/t^{(d-\lambda)/2}$. Note that, at $T=0$, this form should hold for any d, including $d=1$. Preliminary studies [9] of classical Heisenberg spin chains (i.e. $n=3$) confirm the predicted power-law decay of $C(t)$, and yield $\lambda\simeq 0.34$. This can be compared with (3), which gives $\lambda=1/2-(\pi/6\surd 3)(1/n)+O(1/n^2)$ for $d=1$. For $n=3$, this yields $\lambda\simeq 0.40$ to $O(1/n)$. The case $n=2$ (classical XY spins) is found to be anomalous, with a stretched-exponential decay, $C(t)\sim\exp(-a\surd t)$, and a characteristic length scale $L(t)\sim t^{1/4}$ [9]. This can be understood from simple analytical arguments [9], and details will be presented elsewhere.

We thank M A Moore for stimulating discussions. TJN thanks the SERC for financial support.

Note added in proof. Zanetti and Mazenko [10] have determined the exponent λ for $n=\infty$, $d=3$ by solving numerically the equations of the $n=\infty$ limit for a non-zero temperature $T<T_c$. They find $\lambda=3/2$, in agreement with the first term in equation (3).

References

[1] Gunton J D, San Miguel M and Sahni P S 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (New York: Academic) p 267
[2] Furukawa H 1989 *J. Phys. Soc. Japan* **58** 216; 1989 *Phys. Rev.* B **40** 2341
[3] Abramowitz M and Stegun I A (eds) 1965 *Handbook of Mathematical Functions* (New York: Dover)
[4] Janssen H K, Schaub B and Schmittmann B 1989 *Z. Phys.* **73** 539
[5] Coniglio A and Zannetti M 1989 *Europhys. Lett.* **10** 575
[6] Newman T J and Bray A J unpublished
[7] Ma S-K 1976 *Modern Theory of Critical Phenomena* (New York: Benjamin)
[8] Bray A J 1989 *Phys. Rev. Lett.* **62** 2841; 1990 *Phys. Rev.* B, in press
[9] Bray A J, Moore M A and Newman T J unpublished
[10] Zanetti M and Mazenko G F 1987 *Phys. Rev.* B **35** 5043